

# A Doubly Critical Elliptic Problem with Submanifold Singularities

## Résumé

Let  $N \geq 4$ ,  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ , and let  $\Sigma \subset \Omega$  be a smooth closed submanifold of dimension  $k$  with  $2 \leq k \leq N - 2$ . We study the existence of positive solutions  $u \in H_0^1(\Omega)$  to the Euler–Lagrange equation

$$-\Delta u + hu = \lambda \rho_\Sigma^{-s_1} u^{2_{s_1}^* - 1} + \rho_\Sigma^{-s_2} u^{2_{s_2}^* - 1} \quad \text{in } \Omega,$$

where  $h : \Omega \rightarrow \mathbb{R}$  is a continuous potential,  $\lambda > 0$  is a real parameter, and  $0 \leq s_2 < s_1 < 2$ . For  $i = 1, 2$ , the exponents

$$2_{s_i}^* = \frac{2(N - s_i)}{N - 2},$$

correspond to Hardy–Sobolev critical growth, and  $\rho_\Sigma = \text{dist}(\cdot, \Sigma)$  denotes the distance to the submanifold  $\Sigma$ .

The problem involves two Hardy-type singular nonlinearities with different critical exponents, leading to a lack of compactness. Using variational methods, in particular the mountain pass lemma, together with a suitable construction of test functions, we prove existence results under appropriate assumptions. Our analysis shows that the local geometry of  $\Sigma$  and the behavior of the potential  $h$  near  $\Sigma$  play a crucial role in the existence of positive solutions for this doubly critical problem.